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LETTER TO THE EDITOR

Structure of regular representation of quantum universal enveloping algebra $SL_q(2)$ with $q^p = 1$*

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Abstract. In this paper we construct the regular representation quantum Lie algebra of $SL_q(2)$ and its quotient representations and then analyse the reducible structures of these representations when $q^p = 1$ for an integer p. We show how the indecomposable representations of $SL_q(2)$ are obtained by a purely algebraic method.

Due to the development of nonlinear physics relating the Yang–Baxter equation (YBE), quantum group, quantum universal enveloping algebras (QUEA) and their representation theory have drawn considerable attention from both mathematicians and physicists [1, 2]. Besides the standard representation theory of QUEA, [3, 4] the q-deformed boson (oscillator) realization has been proposed independently by different authors [5-7]. Another important development of the representation theory is the discussion of the case with $q^p = 1$ ($p \in \mathbb{D} = \{3, 4, 5, \ldots\}$) [8, 9]. Lusztig first showed that a rich reducible structure will occur in representations when $q^p = 1$ [8], and some indecomposable (reducible but not completely reducible) representations are obtained in this case [10].

In this letter, motivated by Lusztig's work, we construct and analyse some non-simple moduli (representations) of $SL_q(2) \equiv U_q(sl(2))$ when $q^p = 1$. We first build the regular representation of $SL_q(2)$ and then obtain some quotient representations induced by it. These quotient representations are usually indecomposable even if $q^p \neq 1$ ($\forall p \in \mathbb{D}$). In particular, we analyse the reducible structure of these representations when $q^p = 1$ in detail and also show how the finite-dimensional representations are derived from the infinite-dimensional quotient representations. We also suppose that p is an odd integer $\geqslant 3$ without loss of generality [8].

Since $SL_q(2)$ is an associative algebra generated by $J_\pm$ and $K^{\pm 1}$ that satisfies

$$
\begin{aligned}
&[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_3] \\
&[f] \equiv (q^f - q^{-f})(q - q^{-1})^{-1} \qquad q \in \mathbb{C} \text{ (complex field)}
\end{aligned} \tag{1}
$$

we can construct its regular representation on its own linear space by the left transformation action T: $SL_q(2) \rightarrow \mathrm{End}(SL_q(2))$: $T(x) \cdot u = x \cdot u$, $\forall u, x \in SL_q(2)$.

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Using the following relation resulting from (1)

$$\begin{aligned}
&K^{\pm}J_{+}^{n}=q^{\pm 2n}J_{+}^{n}K^{\pm} \qquad K^{\pm}J_{-}^{n}=q^{\mp 2n}J_{-}^{n}K^{\pm}\\
&J_{\pm}K^{n}=q^{\mp 2n}K^{n}J_{\pm} \qquad J_{\pm}(K^{-1})^{n}=q^{\pm 2n}(K^{-1})^{n}J_{\pm}\\
&J_{-}J_{+}^{n}=J_{+}^{n}J_{-+}[n]J_{+}^{n-1}(q^{-n+1}K^{-1}-q^{n-1}K)(q-q^{-1})^{-1}\\
&J_{+}J_{-}^{n}=J_{-}^{n}J_{+}+[n]J_{-}^{n-1}(q^{-n+1}K-q^{n-1}K^{-1})(q-q^{-1})^{-1}
\end{aligned} \tag{2}$$

we explicitly obtain the regular representation

$$\begin{aligned}
&T(J_{+})X(m,n,r)=X(m+1,n,r)\\
&T(J_{-})X(m,n,r)\\
&\qquad =X(m,n+1,r)+\{[m]/(q-q^{-1})\}\\
&\qquad \times\{q^{-m+1+2n}X(m-1,n,r-1)+q^{m-1-2n}X(m-1,n,r+1)\} \qquad (3)\\
&T(K^{\pm 1})X(m,n,r)=q^{\pm 2(m-n)}(X(m,n,r\pm 1).
\end{aligned}$$

On the basis

$$X(m,n,r)=J_{+}^{m}J_{-}^{n}K^{r}=\begin{cases} J_{+}^{m}J_{-}^{n}K^{r} & r=1,2,\ldots\\ 1 & r=0\\ J_{+}^{m}J_{-}^{n}(K^{-1})^{-r} & r=-1,-2\end{cases}$$

$(m, n\in\mathbb{N})$ for $SL_q(2)$ where $\mathbb{N}=\{0,1,2,\ldots\}$.

When $q\to 1$, this representation becomes the master representation of Lie algebra SL(2) [11]. This representation is quite general and some new representations besides the standard ones can be induced on certain quotient spaces by it. We discuss this representation in two cases as follows.

Case 1. $q^p\neq 1$ $(\forall p\in\mathbb{D})$. In this case, because the index n of the vector $X(m,n,r)$ can only be increased by the action of T, all the vectors $X(m,n+N,r)$ for a given $N\in\mathbb{N}$ span an invariant subspace V_N and so the representation T is reducible. However, there does not exist an invariant complementary space for V_N, i.e. the representation T is indecomposable (reducible but not completely reducible). For the sequence of T-invariant subspaces

$$SL_q(2)=V_0\supset V_1\supset V_2\supset\ldots\supset V_N\supset V_{N+1}\supset\ldots$$

the representation T induces various quotient representations of $SL_q(2)$ $T^{[N,N']}$ on each quotient space $V_N/V_{N'}$ $(N'>N)$. Let $T^{[N]}=T^{[N,N+1]}$, then T has a semidirect sum structure, $T=T^{[0]}\bar{\oplus}\ldots T^{[1]}\bar{\oplus}\ldots\bar{\oplus}T^{[N]}\bar{\oplus}\ldots$.

Case 2. $q^p=1$ $(p\in\mathbb{D})$, $[\alpha p]=0$ for $\alpha\in\mathbb{N}$

$$T(J_{-})X(\alpha p,n,r)=X(\alpha p,n+1,r)$$

i.e. $X(\alpha p,n,r)$ $(n,r\in\mathbb{N})$ are extreme vectors and they define an T-invariant subspace V_0^{α}: $\{X(\alpha p+m,n,r)|m,n,r\in\mathbb{N}\}$. Then, we have some 'smaller' T-invariant subspaces

$$V_N^{\alpha}=V_0^{\alpha}\cap V_N=\{X(\alpha p+m,N+n,r)|m,n,r\in\mathbb{N}\}$$

and 'bigger' ones

$$B_N^{\alpha}=V_0^{\alpha}\cup V_N=\{X(\alpha p+m,n,r),X(m',n'+N,r')|m,n,r,m',n',r'\in\mathbb{N}\}.$$

From these subspaces, we naturally obtain some new subrepresentations.

Now we consider a quotient representation Γ of $SL_q(2)$ with definite weight induced by the regular representation T.

Corresponding to a left deal $I=\{X(K-q^{2\lambda})|\text{ all } x\in SL_q(2)\}$ for $\lambda\in\mathbb{C}$, the quotient space $Q=SL_q(2)/I$ has a basis

$$X(m,n)=X(m,n,O)\text{ Mod } I\qquad m,n\in\mathbb{N}.$$

An easy calculation allows us to obtain a quotient representation on the quotient space

$$\begin{aligned}&\Gamma(J_+)X(m,n)=X(m+1,n)\\&\Gamma(J_-)X(m,n)=X(m,n+1)+[m](2n-m+1-2\lambda]X(m-1,n)\\&\Gamma(K^{\pm})X(m,n)=q^{\pm2(m-n+\lambda)}X(m,n).\end{aligned}\tag{4}$$

Like the regular representation T on $V_0\equiv SL_q(2)$, this representation is also indecomposable. Its reducible structure can be induced by the reducible structure of T. Consider a projection mapping φ: $V_0=SL_q(2)\to Q$ such that

$$\varphi(X(m,n,r))=q^{2r\lambda}X(m,n)=q^{2r\lambda}X(m,n,0)\text{ Mod } L.$$

This mapping defines the quotient representation (4): $\Gamma=\varphi\cdot T\cdot\varphi^{-1}$. If S is a T-invarient subspace of V_0, then $\varphi(S)=\{\varphi(x)|\text{ all } x\in V_0\}$ is a Γ-invariant subspace of Q. Therefore, we have Γ-invariant subspaces of Q corresponding to V_N, V_0^α, V_N^α and B_α^N respectively

$$\begin{aligned}&Q_N=\varphi(V_N)\colon\{X(m,n+N)|m,n\in\mathbb{N}\}\\&Q_0^\alpha=\varphi(V_0^\alpha)\colon\{X(\alpha p+m,n)|m,n\in\mathbb{N}\}\\&Q_N^\alpha=\varphi(V_N^\alpha)\colon\{X(\alpha p+m,n+N)|m,n\in\mathbb{N}\}\\&W_N^\alpha=\varphi(B_\alpha^N)\colon\{X(m,n+N),X(\alpha p+m',n')|m,n,m',n'\in\mathbb{N}\}.\end{aligned}$$

There is also a sequence of Γ-invariant subspaces

$$Q=Q_0\supset Q_1\supset Q_2\supset\ldots\supset Q_N\supset Q_{N+1}\supset\ldots$$

and the quotient representations $\Gamma^{[N,N']}$ on $Q_N/Q_{N'}$ $(N'>N)$.

Although all the subrepresentations on the subspaces Q_N, Q_0^α, Q_N^α and W_N^α are infinite dimensional, a class of quotient spaces $Q_N^\alpha=Q/W_N^\alpha$: $\{\tilde{X}(m,n)=X(m,n)\text{ Mod } W_n^\alpha|n\in\mathbb{N}\}$ is finite dimensional and the dimension is

$$\dim\cdot Q_{R\alpha}=\alpha Np.$$

On the quotient space Q_N^α, the representation $\Gamma^{[N,\alpha]}$ induced by Γ is formally defined by (4) and the additional requirements

$$\begin{aligned}&\Gamma^{[N,\alpha]}(J_+)X(\alpha p-1,n)=0\\&\Gamma^{[N,\alpha]}(J_-)X(m,N-1)=0.\end{aligned}\tag{5}$$

For example, when $\alpha=1$ and $N=2$, we have a $2p-D$ representation

$$\begin{aligned}&\Gamma^{[2,1]}(J_+)\tilde{X}(m,n)=\tilde{X}(m+1,n)\qquad m=0,1,2,\ldots,p-2\\&\Gamma^{[2,1]}(J_+)\tilde{X}(p-1,n)=0\\&\Gamma\varepsilon\varsigma\phi\varsigma^{[2,1]}(J_-)\tilde{X}(m,0)=\tilde{X}(m,1)+[m][1-m-2\lambda]\tilde{X}(m-1,0)\\&\Gamma^{[2,1]}(J_+)\tilde{X}(m,1)=[m][3-m-2\lambda]\tilde{X}(m-1,n)\\&\Gamma^{[2,1]}(K^{\pm})\tilde{X}(m,n)=q^{\pm2(m-n+\lambda)}\tilde{X}(m,n).\end{aligned}$$

It is easy to check that $\Gamma^{[2,1]}$ as well as Γ and T forms a representation of $SL_q(2)$.

In the last paragraph the representations discussed were defined on the basis $X(m,n)$ or $\tilde{X}(m,n)$ with two indices m and n. Now we discuss the representations whose bases have one index.

Let $L=\{x(J_+-\mu\mathbb{1})|$ all $x\in Q\}$ be a left ideal of Q for $\mu\in\mathbb{C}$ and $R=Q/L$: $\{x(n)=X(n,0)$ Mod $L|n\in\mathbb{N}\}$ be the corresponding quotient space. On the quotient space R, the representation ρ induced by Γ is

$$
\begin{aligned}
&\rho(J_+)X(n)=\mu X(n)+[n][2\lambda-n+1]X(n-1)\\
&\rho(J_-)X(n)=X(n+1)\\
&\rho(K^{\pm1})X(n)=q^{\pm2(\lambda-n)}X(n).
\end{aligned}
\tag{7}
$$

Because the basic relations (1) allow a difference of J_+ by a $\mathbb{C}$-number μ, i.e. $J'_+=J_+-\mu$; J_- and J_3 still satisfy the relations (1), we can let $\mu=0$ without loss of generality and

$$
\begin{aligned}
&\rho(J_+)X(n)=[n][2\lambda-n+1]X(n-1)\\
&\rho(J_-)X(n)=X(n+1)\\
&\rho(K^{\pm1})X(n)=q^{\pm2(\lambda-n)}X(n)
\end{aligned}
\tag{8}
$$

is also a representation of $\mathrm{SL}_q(2)$. In fact, the above representation is of a Verma module with highest weight 2λ. The simple modulus relating to this Verma module has been well classified in [3, 4] and [8, 9] for both cases, $q^p\neq1$ ($\forall p\in\mathbb{D}$) and $qp'=1$ ($P'\in\mathbb{D}$). Here, we will pay attention to the case of non-simple modulus with $q^p=1$ for a given p.

(1) In the case with non-integer $2\lambda\in\mathbb{C}$, the representation (8) is irreducible unless $q^p=1$ ($p\in\mathbb{D}$). When $q^p=1$, $X(\alpha p)$ for $\alpha\in\mathbb{N}$ is an extreme vector such that $\rho(J_+)X(\alpha p)=0$. Then, we have a ρ-invariant subspace W_α: $\{X(\alpha p+n)|n\in\mathbb{N}\}$. On the quotient space $R_\alpha=R/W_\alpha$: $\{\tilde{X}(n)=X(n)$ Mod $W\alpha|n\in\mathbb{N}\}$, ρ induces a αp-dimensional representation $\rho^{[\alpha]}$:

$$
\begin{aligned}
&\rho^{[\alpha]}(J_+)\tilde{X}(n)=[n][2\lambda-n+1]\tilde{X}(n-1)\\
&\rho^{[\alpha]}(J_-)\tilde{X}(n)=\tilde{X}(n+1) \qquad n=0,1,2,\ldots,\alpha p-1\\
&\rho^{[\alpha]}(J_-)\tilde{X}(n=\alpha p-1)=0\\
&\rho^{[\alpha]}(K^{\pm1})\tilde{X}(n)=q^{\pm2(\lambda-n)}\tilde{X}(n).
\end{aligned}
\tag{9}
$$

Because $W_{\alpha'}$ is an invariant subspace of W_α when $\alpha'>\alpha$, the representation $\rho^{[\alpha]}(\alpha>1)$ is still reducible. As Roche and Araudo have proved [9], the quotient representation $\rho^{[\alpha,\alpha+1]}$:

$$
\begin{aligned}
&\rho^{[\alpha,\alpha+1]}(J_+)Y(n)=[n][2\lambda-n+1]Y(n-1)\\
&\rho^{[\alpha,\alpha+1]}(J_-)Y(n)=Y(n+1) \qquad n=0,1,2,\ldots,p^{-1}\\
&\rho^{[\alpha,\alpha+1]}(J_-)Y(p-1)=0\\
&\rho^{[\alpha,\alpha+1]}(K^{\pm1})Y(n)=q^{\pm2(\lambda-n)}Y(n)
\end{aligned}
\tag{10}
$$

induced by $\rho^{[\alpha]}$ on the quotient space $R_\alpha/R_{\alpha+1}=\{Y(n)=\tilde{X}(n)$ Mod $R\alpha+1|\ n=0,1,2,\ldots,p-1\}$ is irreducible.

(2) In the case where $2\lambda\in\mathbb{N}$ and $q^p=1$ ($p\in\mathbb{D}$), besides the invariant subspaces W_α, there are ρ-invariant subspaces S_λ: $\{X(2\lambda+1+n)|n\in\mathbb{N}\}$ determined by the vectors

$X(2\lambda+1)$ such that $\rho(J_+)X(2\lambda+1)=0$. On the quotient space $R/S_\lambda \equiv R^\lambda$: $\{u(n)=X(n) \bmod S_\lambda | n\in\mathbb{N}\}$ a representation ρ_λ with dimension $(2\lambda+1)$ is induced by (8)

$$\begin{aligned}
&\rho_\lambda(J_+)u(n)=[n][2\lambda+n+1]u(n-1)\\
&\rho_\lambda(J_-)u(n)=u(n+1) \qquad n=0,1,2,\ldots,2\lambda-1\\
&\rho_\lambda(J_-)u(2\lambda)=0\\
&\rho_\lambda(K^\pm)u(2\lambda)=q^{\pm 2(\lambda-n)}u(2\lambda).
\end{aligned} \tag{11}$$

(a) When $2\lambda+1\leqslant p$, the representation defined by (11) is irreducible. This case of simple modulus has been discussed in [9].

(b) When $2\lambda+1>\alpha p$ (for a $\alpha\in\mathbb{N}$), the representation (11) is not irreducible. In fact, S_λ is a ρ-invariant subspace of W_α and $R^\lambda=R/S_\lambda$ has ρ_λ-invariant subspaces R^λ: $\{u(\beta p+k)=\tilde{u}(k)|k=0,1,2,\ldots,2\lambda-\beta p\}$ $(\beta=1,2,\ldots,\alpha)$. On a subspace R^λ_β, we have a subrepresentation ρ^λ_β:

$$\begin{aligned}
&\rho^\lambda_\beta(J_+)\tilde{u}(k)=[\beta p+k][2\lambda-\beta p-k+1]\tilde{u}(k-1)\\
&\rho^\lambda_\beta(J_-)\tilde{u}(k)=\tilde{u}(k+1)\\
&\rho^\lambda_\beta(J_-)\tilde{u}(2\lambda-\beta k)=0\\
&\rho^\lambda_\beta(K^\pm)\tilde{u}(k)=q^{\pm 2(\lambda-k)}\tilde{u}(k).
\end{aligned} \tag{12}$$

However, R^λ_β has no invariant complementary subspace and the representation ρ_λ given by (11) is indecomposable.

It is easy from (2) to observe that $(K^{\pm p}, X]=[J^p_\pm, X]=0$, $\forall x\in \mathrm{SL}_q(2)$, that is to say, $J^p_\pm$ and $K^{\pm p}$ are the elements in the centre of $\mathrm{SL}_q(2)$. On a certain space R, we also prove that J^p_+ actually vanishes. In fact, when $n<p$, $\Gamma(J^p_+)X(n=)0$; when $n=\alpha p+n'>p$: where $0\leqslant n'<p$, $\alpha\geqslant 1$

$$\begin{aligned}
\Gamma(J^p_+)X(n)&=[n][n-1][n-2]\ldots[n-p+1][n-p]\\
&\quad\times[2\lambda-n+1]\ldots[2\lambda-n+p]X(n-p)\\
&=[\alpha\rho][\rho-1]![2\lambda-n+p]!([2\lambda-n]!)^{-1}X(n-\rho).
\end{aligned}$$

Then, we can introduce Lusztig operator

$$\hat{L}=\lim_{q^p\to 1}\frac{J^p_+}{[p]!} \tag{13}$$

that acts on $X(n)$ as

$$\hat{L}X(n)=\begin{cases} 0 & \text{if } n\leqslant p\\ \alpha\left(\dfrac{[2\lambda-n+p]!}{[2\lambda-n]!}\right)X_{[n-\rho)} & \text{if } n>\rho. \end{cases} \tag{14}$$

Now we consider the representations of an extension $\hat{\mathrm{S}}\mathrm{L}_q(2)$ of $\mathrm{SL}_q(2)$ by adding the Lusztig element $\hat{L}$. The representations (8) and (11) of $\mathrm{SL}_q(2)$ are still the representations of $\hat{\mathrm{S}}\mathrm{L}_q(2)$. However, because W_α and R^λ_β are not invariant under the action of $\hat{L}$, the representation (8) is irreducible when $2\lambda\neq$ integer and the representation (11) ρ_λ is also irreducible for given $\lambda\in\mathbb{N}$.

Finally, it should be pointed out that the main results of this letter are not covered by previous work [8, 9] because we were mainly involved with the cases of non-simple moduli relating to the regular representation.

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